ON THE COLLISION OF GASEOUS JETS

(O SOUDARENII GAZOVYKH STRUI)

PMM Vol.26, No.2, 1962, pp. 308-315

N.N. MAKEEV
(Perm')

(Received October 26, 1961)

An exact solution is presented of the plane problem of the collision of gaseous jets issuing from coaxial channels of finite width with parallel walls. By using the theory of gaseous jets with subsonic velocities, the problem is reduced to a boundary value problem for Chaplygin's equation, the solution of which is presented in the form of Fourier series.

Chaplygin [1] proposed a method of solution of the problem of subsonic jet flow in the case when there is only one specified characteristic velocity. Chaplygin's method was extended by Fal'kovich [2] for the class of problems with more than one characteristic velocity. This extension allows us to investigate a number of questions in the theory of colliding jets, which are of interest in connection with the improvement of the theory of a combined jet, developed in the first approximation by Lavrent'ev [3].

1. Let us consider the collision of gaseous jets issuing from coaxial channels of finite width with parallel walls, and with the subsonic velocity v_0 on the free surfaces, into a gaseous medium at rest. In view of the symmetry of the problem it is sufficient to consider only a half of the region of flow (Fig. 1) with one branch of the jet.

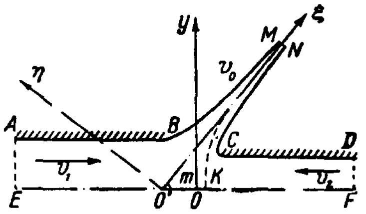

Fig. 1.

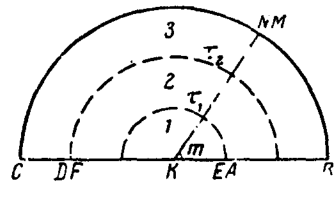

Fig. 2.

Here AB and CD are the walls of the channels, and BM and CN are the free surfaces of the jet, whilst δ is the contact surface between the jet streams, with the point of branching at K. Let v_1 and v_2 be the velocities of the gas at the sections at infinity AE and DF in the channels, d_1 and d_2 the diameters of the channels, d_0 the width of the branch of the jet at infinity, m the angle between the x-axis and the velocity vector at infinity in the jet, whilst $2h$ is the distance between the outlet orifices of the channels.

We shall assume that on the streamline EOF the stream function $\psi = 0$. Then, if the discharge of gas across the sections AE and DF be denoted by $1/2\ Q_1$ and $1/2\ Q_2$ respectively, and the discharge of gas across the section MN by $1/2\ Q_0$, then $\psi = 1/2\ Q_1$ on the streamline ABM and $\psi = -1/2\ Q_2$ on the streamline DCN.

Let us denote by v the velocity, v_{max} the maximum velocity of flow, and θ the angle formed by the velocity vector with the x-axis. Then, transforming to the variables θ and $\tau = v^2/v^2_{max}$, we obtain in the velocity hodograph plane the region of flow depicted in Fig. 2, where the semi-circle CB corresponds to $\tau = \tau_0$.

The boundary conditions have the form

$$\begin{aligned}
&\psi = 0 && \text{when } \theta = 0, && 0 < \tau < \tau_1 \\
&\psi = \tfrac{1}{2} Q_1 && \text{when } \theta = 0, && \tau_1 < \tau < \tau_0 \\
&\psi = 0 && \text{when } \theta = \pi, && 0 < \tau < \tau_2 \\
&\psi = -\tfrac{1}{2} Q_2 && \text{when } \theta = \pi, && \tau_2 < \tau < \tau_0
\end{aligned} \tag{1.1}$$

$$\begin{aligned}
&\psi = \tfrac{1}{2} Q_1 && \text{when } \tau = \tau_0, && 0 < \theta < m \\
&\psi = -\tfrac{1}{2} Q_2 && \text{when } \tau = \tau_0, && m < \theta < \pi
\end{aligned} \tag{1.2}$$

Accordingly, the solution of the problem posed reduces to finding the solution of the internal problem of Dirichlet for Chaplygin's equation

$$4\tau^2(1-\tau)\frac{\partial^2\psi}{\partial\tau^2} + 4\tau[1+(\beta-1)\tau]\frac{\partial\psi}{\partial\tau} + [1-(2\beta+1)\tau]\frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{1.3}$$

in a semi-circular domain. Here $\beta = 1/(\gamma - 1)$, $\gamma = c_p/c_v$.

The solution of the problem will be sought in the form

$$\psi_1 = \sum_{n=1}^{\infty} a_n Z_{n/2}(\tau) \sin n\theta \tag{1.4}$$

$$\psi_2 = \frac{1}{2} Q_1 \frac{\pi - \theta}{\pi} + \sum_{n=1}^{\infty} \{A_n Z_{n/2}(\tau) + B_n \zeta_{n/2}(\tau)\} \sin n\theta \tag{1.5}$$

$$\psi_3 = \frac{1}{2}\left(Q_1 - Q_0 \frac{\theta}{\pi}\right) + \sum_{n=1}^{\infty} \{C_n Z_{n/2}(\tau) + D_n \zeta_{n/2}(\tau)\} \sin n\theta \tag{1.6}$$

Here the suffix for ψ corresponds to the number of the subregion of the semi-circle, in which the given solution is sought, $Z_{n/2}(\tau)$ is the integral of the equation

$$4\tau^2(1-\tau) Z''_{n/2} + 4\tau[1 + (\beta - 1)\tau] Z'_{n/2} - n^2[1 - (2\beta + 1)\tau] Z_{n/2} = 0 \tag{1.7}$$

which is regular at $\tau = 0$, considered by Chaplygin [1]; $\zeta_{n/2}(\tau)$ is Cherry's function [4], which is the second linearly independent integral of Equation (1.7), considered by Fal'kovich [2]. The coefficients a_n, A_n, B_n, C_n, D_n are yet to be determined.

The stream functions determined by means of (1.4) to (1.6) satisfy the boundary conditions (1.1). We need now to fulfil the following conditions.

1. The function ψ_3 must satisfy the conditions (1.2).

2. The function ψ_2 must be the analytic continuation of ψ_1 from the region (1) into the region (2), whilst ψ_3 is the analytic continuation of ψ_2 from region (2) into the region (3):

$$\begin{aligned} &\psi_1 = \psi_2, \quad \frac{\partial \psi_1}{\partial \tau} = \frac{\partial \psi_2}{\partial \tau} \quad \text{when } \tau = \tau_1 \\ &\psi_2 = \psi_3, \quad \frac{\partial \psi_2}{\partial \tau} = \frac{\partial \psi_3}{\partial \tau} \quad \text{when } \tau = \tau_2 \end{aligned} \qquad (0 < \theta < \pi) \tag{1.8}$$

Conditions (1.2) and (1.8) reduce to the system of equations

$$\begin{aligned} C_n Z_{n/2}(\tau_0) + D_n \zeta_{n/2}(\tau_0) &= -\frac{Q_0}{n\pi} \cos nm \\ (A_n - a_n) Z_{n/2}(\tau_1) + B_n \zeta_{n/2}(\tau_1) &= -\frac{Q_1}{n\pi} \\ (A_n - a_n) Z'_{n/2}(\tau_1) + B_n \zeta'_{n/2}(\tau_1) &= 0 \\ (A_n - C_n) Z_{n/2}(\tau_2) + (B_n - D_n) \zeta_{n/2}(\tau_2) &= (-1)^n \frac{Q_2}{n\pi} \\ (A_n - C_n) Z'_{n/2}(\tau_2) + (B_n - D_n) \zeta'_{n/2}(\tau_2) &= 0 \end{aligned}$$

Since the Wronskian is

$$W_{n/2}(\tau) = W\{Z_{n/2}(\tau), \zeta_{n/2}(\tau)\} = \frac{n}{2\tau}(1-\tau)^\beta$$

then the ultimate solution of the problem will have the form

$$\psi_1 = \frac{Q_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} f_{n/2}^{(1)}(\tau) \sin n\theta \tag{1.9}$$

$$\psi_2 = \frac{1}{2} Q_1 \frac{\pi - \theta}{\pi} + \frac{Q_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} f_{n/2}^{(2)}(\tau) \sin n\theta \tag{1.10}$$

$$\psi_3 = \frac{1}{2}\left(Q_1 - Q_0 \frac{\theta}{\pi}\right) + \frac{Q_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} f_{n/2}(\tau) \sin n\theta \tag{1.11}$$

Here

$$f_{n/2}^{(1)}(\tau) = \left[-\cos nm + \frac{2\sigma_1\tau_1}{n(1-\tau_1)^\beta} T'_{n/2}(\tau_1, \tau_0) + \right.$$
$$\left. + (-1)^n \frac{2\sigma_2\tau_2}{n(1-\tau_2)^\beta} T'_{n/2}(\tau_2, \tau_0)\right] \frac{Z_{n/2}(\tau)}{Z_{n/2}(\tau_0)}$$

$$f_{n/2}^{(2)}(\tau) = \frac{\sigma_1}{(1-\tau_1)^\beta} \frac{Z_{n/2}(\tau_1)}{Z_{n/2}(\tau_0)} x_{n/2}(\tau_1) T_{n/2}(\tau, \tau_0) +$$
$$+ \left[-\cos nm + (-1)^n \frac{2\sigma_2\tau_2}{n(1-\tau_2)^\beta} T'_{n/2}(\tau_2, \tau_0)\right] \frac{Z_{n/2}(\tau)}{Z_{n/2}(\tau_0)}$$

$$f_{n/2}(\tau) = -\frac{Z_{n/2}(\tau)}{Z_{n/2}(\tau_0)} \cos nm + \left[\frac{\sigma_1}{(1-\tau_1)^\beta} \frac{Z_{n/2}(\tau_1)}{Z_{n/2}(\tau_0)} x_{n/2}(\tau_1) + \right.$$
$$\left. + (-1)^n \frac{\sigma_2}{(1-\tau_2)^\beta} \frac{Z_{n/2}(\tau_2)}{Z_{n/2}(\tau_0)} x_{n/2}(\tau_2)\right] T_{n/2}(\tau, \tau_0)$$

where

$$T_{n/2}(\tau, \tau_0) = Z_{n/2}(\tau) \zeta_{n/2}(\tau_0) - \zeta_{n/2}(\tau) Z_{n/2}(\tau_0)$$

$$T'_{n/2}(\tau_\nu, \tau_0) = [T'_{n/2}(\tau, \tau_0)]_{\tau=\tau_\nu}, \qquad \sigma_1 = Q_1/Q_0, \qquad \sigma_2 = Q_2/Q_0$$

$$x_{n/2}(\tau) = x_{n/2}(\tau) = \frac{2\tau}{n} \frac{Z'_{n/2}(\tau)}{Z_{n/2}(\tau)} \text{ — Chaplygin's function}$$

For later use, we note that

$$T'_{n/2}(\tau_\nu, \tau_\nu) = W_{n/2}(\tau_\nu), \qquad f_{n/2}(\tau_0) = -\cos nm$$

$$f'_{n/2}(\tau_0) = \frac{n}{2\tau_0}\left\{-x_{n/2}(\tau_0) \cos nm + \sigma_1 \left(\frac{1-\tau_0}{1-\tau_1}\right)^\beta \frac{Z_{n/2}(\tau_1)}{Z_{n/2}(\tau_0)} x_{n/2}(\tau_1) + \right.$$
$$\left. + (-1)^n \sigma_2 \left(\frac{1-\tau_0}{1-\tau_2}\right)^\beta \frac{Z_{n/2}(\tau_2)}{Z_{n/2}(\tau_0)} x_{n/2}(\tau_2)\right\} \tag{1.12}$$

From solutions (1.9) to (1.12), as particular cases, there follows a number of well-known results. For example, from (1.9) with $m = \pi/2$, $\tau_1 = \tau_2 = \tau_0$, $d_1 = d_2$ there follows Slezkin's result [5]. We can also obtain Troshin's [6,7] and Chaplygin's results [1].

2. Let us introduce a system of coordinates ξ, η with center at the point $O'(-a, 0)$; the axis of ξ will make an angle m with the axis of x and instead of the angle θ we shall consider the angle $\vartheta = \theta - m$. In what follows we are concerned with the stream function ψ_3. In the chosen system of coordinates we shall have [2]

$$\eta = \frac{(1-\tau)^{-\beta}}{v}\int\left(2\tau\frac{\partial\psi}{\partial\tau}\sin\vartheta + \frac{\partial\psi}{\partial\vartheta}\cos\vartheta\right)d\vartheta + \eta_0(\tau) \qquad (2.1)$$

where $\eta_0(\tau)$ is an arbitrary function.

Let us determine the coefficient of contraction of the jet. By the coefficient of contraction K of an unsymmetrical jet we mean the ratio of the minimum width of the jet to the value of the projection of the width of the aperture BC on the axis of η

$$K = \frac{d_0}{2h\sin m} \qquad (2.2)$$

Immediately from Fig. 1 we obtain the coordinates η_B and η_C of the points B and C:

$$\eta_B = (a+h)\sin m + \tfrac{1}{2}d_1\cos m, \qquad \eta_C = (a-h)\sin m + \tfrac{1}{2}d_2\cos m \qquad (2.3)$$

On the other hand, the coordinates of the points B and C can be determined from (2.1). Setting $\tau = \tau_0$ and integrating (2.1) along the intervals $[0, -m]$ and $[0, \pi - m]$ allowing for (1.12), we obtain η_B and η_C respectively. Since $\eta = 0$ when $\vartheta = 0$, then $\eta_0(\tau) = 0$.

Determining the quantity $2h \sin m$ from (2.3) and making use of the relation

$$\eta_B - \eta_C = \frac{Q_0\sin m}{2v_0(1-\tau_0)^\beta}\left(\frac{8\tau_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}f'_n(\tau_0) + \sin m\right)$$

we obtain with the help of (2.2) a formula for the coefficient of contraction of the jet

$$\frac{1}{K} = \sin m\left(\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4n}{4n^2-1}\chi_n'(\tau_0) + \sin m\right) - \frac{d_1-d_2}{2d_0}\cos m \qquad (2.4)$$

where

$$\chi_n'(\tau_0) = -x_n(\tau_0)\cos 2nm + \sigma_1\left(\frac{1-\tau_0}{1-\tau_1}\right)^\beta\frac{Z_n(\tau_1)}{Z_n(\tau_0)}x_n(\tau_1) + \sigma_2\left(\frac{1-\tau_0}{1-\tau_2}\right)^\beta\frac{Z_n(\tau_2)}{Z_n(\tau_0)}x_n(\tau_2)$$

In addition to this formula we need the equation of continuity

$$Q_1 + Q_2 = Q_0$$

which can be reduced to the form

$$2d_0 = \left(\frac{\tau_1}{\tau_0}\right)^{1/2}\left(\frac{1-\tau_1}{1-\tau_0}\right)^{\beta} d_1 + \left(\frac{\tau_2}{\tau_0}\right)^{1/2}\left(\frac{1-\tau_2}{1-\tau_0}\right)^{\beta} d_2 \qquad (2.5)$$

Let us consider a number of particular cases arising from Formula (2.4). In the case of infinitely wide channels ($\tau_1 = \tau_2 = 0$) we have

$$\frac{1}{K} = \sin m \left(\sin m - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4n}{4n^2-1}x_n(\tau_0)\cos 2nm\right)\left(1 + \frac{d_1 - d_2}{4h}\cot m\right)^{-1} \qquad (2.6)$$

Here the difference $d_1 - d_2$ has to be interpreted as the distance between the points B and C along the y-axis.

When $m = \pi/2$ (2.6) leads to Chaplygin's formula [1]

$$\frac{1}{K} = 1 - \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{4n^2-1}x_n(\tau_0)$$

When $\tau_1 = \tau_2 = \tau_0$ we obtain the case of collision of free jets. In this case (2.4) takes the form

$$\frac{1}{K} = \sin m\left(\sin m + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4n}{4n^2-1}(1-\cos 2nm)\,x_n(\tau_0)\right) - \frac{1-k}{1+k}\cos m \quad \left(k = \frac{d_2}{d_1}\right)$$

When $m = \pi/2$, $\tau_2 = \tau_1$, $d_2 = d_1$ we obtain the coefficient K for the problem of the efflux of gas from a vessel, considered by Troshin [7]. The formula in this case is

$$\frac{1}{K} = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4n}{4n^2-1}x_n(\tau_0) + \frac{2}{\pi}\left(\frac{1-\tau_0}{1-\tau_1}\right)^{\beta}\sum_{n=1}^{\infty}\frac{4n}{4n^2-1}\frac{Z_n(\tau_1)}{Z_n(\tau_0)}x_n(\tau_1)$$

The case when $m = \pi/2$, $\tau_1 = \tau_2 = \tau_0 < \tau_s = 1/(2\beta + 1)$, $d_1 = d_2$ corresponds to the problem of head-on collision of two gaseous jets of the same width, considered by Slezkin [5].

In the case of incompressible fluid (2.4) takes the form

$$\frac{1}{K} = \sin m\left\{\sin m + \frac{2}{\pi}\left[\frac{1}{1+k\lambda}\left(\frac{v_1}{v_0}+\frac{v_0}{v_1}\right)\tanh^{-1}\frac{v_1}{v_0} + \right.\right.$$

$$\left.\left. + \frac{k\lambda}{1+k\lambda}\left(\frac{v_2}{v_0}+\frac{v_0}{v_2}\right)\tanh^{-1}\frac{v_2}{v_0} + \cos m \ln\tan\frac{m}{2}\right]\right\} - \frac{d_1-d_2}{2d_0}\cos m \qquad (2.7)$$

Here $\lambda = v_2/v_1$, whilst d_0 is determined from the equation of continuity

$$v_1 d_1 + v_2 d_2 = 2 v_0 d_0$$

When $m = \pi/2$, $k = \lambda = 1$, we obtain from (2.7) the Expression [7]

$$\frac{1}{K}=1+\frac{2}{\pi}\left(\frac{v_1}{v_0}+\frac{v_0}{v_1}\right)\tanh^{-1}\frac{v_1}{v_0} \tag{2.8}$$

When $1/\lambda = \tan(\nu/2)$ this expression reduces to the form indicated by Zhukovskii [8]. When $v_0 \to \infty$ we obtain from (2.8) Kirchhoff's formula

$$\frac{1}{K}=1+\frac{2}{\pi}$$

Let us turn to consideration of the Formula (2.6). In the case when $\tau_0 = \tau_s$, we can make use of Frankl's asymptotic expansion [9] for Chaplygin's functions:

$$x_n(\tau_s)=\frac{C_0}{(2n)^{1/3}}+\frac{C_1}{2n}+\frac{C_2}{(2n)^{5/3}}+\frac{C_3}{(2n)^{7/3}}+O\left(\frac{1}{n^3}\right) \tag{2.9}$$

The series obtained after substitution of (2.9) in (2.6) fail to give rapid convergence. For this we make use of Lindelöf's formula [10]

$$\sum_{n=1}^{\infty}\frac{x^n}{n^s}=\Gamma(1-s)(-\ln x)^{s-1}+\sum_{n=0}^{\infty}\zeta(s-n)\frac{(\ln x)^n}{n!} \tag{2.10}$$

valid in the complex plane of x with a cut along the real axis from 1 to ∞, when $\operatorname{Re} S > 1$ and S is not an integer. Here $\Gamma(t)$ is Euler's function $\zeta(t)$ is Riemann's function.

In the particular case when $|x| = 1$, Lindelöf's formula has the form

$$\sum_{n=1}^{\infty}\frac{\cos n\theta}{n^s}=+\Gamma(1-s)\theta^{s-1}\sin\frac{s\pi}{2}+\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(s-2n)}{(2n)!}\theta^{2n} \tag{2.11}$$

$$\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^s}=\Gamma(1-s)\theta^{s-1}\cos\frac{s\pi}{2}+\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(s-2n-1)}{(2n+1)!}\theta^{2n+1} \tag{2.12}$$

The principal terms of these expansions coincide, apart from the sign, with the expressions derived by Zigmund [1] and Fal'kovich [12]. The series in (2.11) and (2.12) converge absolutely when $|\theta| < 2\pi$.

Using the identity

$$\frac{1}{4n^2-1}=\frac{1}{4n^2}+\frac{1}{16n^4}+\frac{1}{16n^4(4n^2-1)} \tag{2.13}$$

and substituting (2.11) and (2.13) in (2.6), we obtain approximately

$$-\frac{1}{K}=\sin m\left\{(1+c_1)\sin m+2^{-4/3}\left(c_0+c_2 2^{-4/3}+\frac{1}{4}c_3\right)\frac{\cos 2m}{3\pi}-\right.$$
$$\left.-2\varphi(m)+\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n u_n}{(2n)!}(2m)^{2n}\right\}\left[1+\frac{d_1-d_2}{4h}\operatorname{ctg} m\right]^{-1} \tag{2.14}$$

where

$$\varphi(m) = -\frac{1}{\pi}c_1 + c_0\frac{m^{1/3}}{\Gamma(4/3)} + c_2\frac{m^{5/3}}{\Gamma(8/3)} - (c_0 + c_3)\frac{m^{7/3}}{\Gamma(10/3)} - c_2\frac{m^{11/3}}{\Gamma(14/13)} + c_3\frac{m^{13/3}}{\Gamma(16/3)}$$

$$u_n = c_0 2^{-1/3}\zeta\left(\frac{4}{3} - 2n\right) + c_2 2^{-5/3}\zeta\left(\frac{8}{3} - 2n\right) + (c_0 + c_3)\,2^{-7/3}\times$$
$$\times\,\zeta\left(\frac{10}{3} - 2n\right) + c_2 2^{-11/3}\zeta\left(\frac{14}{3} - 2n\right) + c_3 2^{-13/3}\zeta\left(\frac{16}{3} - 2n\right)$$

The coefficients in (2.9) were determined by Fal'kovich [13]:

$$c_0 = -(\varkappa + 1)^{1/3}\mu'(0), \qquad c_1 = -\frac{2\varkappa + 5}{10}, \qquad c_2 = \frac{24\varkappa^2 + 70\varkappa + 85}{140(\varkappa + 1)^{1/3}}[\mu'(0)]^2$$

whilst for the coefficient c_3 computations give the following value

$$c_3 = -\frac{1343\varkappa^3 - 19530\varkappa^2 + 3990\varkappa + 17150}{18900(\varkappa + 1)^{2/3}}\mu'(0)$$

where

$$\mu'(0) = -3^{1/3}\frac{\Gamma(2/3)}{\Gamma(1/3)} = -0.72898$$

It remains to determine the angle of departure of the jet m, appearing in the formula for the coefficient of contraction. Taking for the control surface the contour bounding the region of flow (Fig. 1), and applying Euler's well-known theorem, we obtain the relation

$$d_1(p_1 + \rho_1 v_1^2) - d_2(p_2 + \rho_2 v_2^2) = 2d_0(p_0 + \rho_0 v_0^2)\cos m \qquad (2.15)$$

expressing the law of momentum.

Here the indices 1, 2 and 0 relate to values at the sections AE, DF and MN, respectively. Making use of the relations

$$p_n = p^\circ(1 - \tau_n)^{\beta+1}, \qquad \rho_n = \rho^\circ(1 - \tau_n)^{\beta} \qquad (n = 1, 2)$$

where p°, ρ° are the stagnation parameters of the gas, and the equation of continuity (2.5), we reduce (2.15) to the form

$$\cos m = \left(\frac{\tau_0}{\tau_1}\right)^{\frac{1}{2}}\frac{1 + (2\beta + 1)\tau_1}{1 + (2\beta + 1)\tau_0}\;\frac{1 - k\left(\dfrac{1 - \tau_2}{1 - \tau_1}\right)^{\beta}\dfrac{1 + (2\beta + 1)\tau_2}{1 + (2\beta + 1)\tau_1}}{1 - k\left(\dfrac{\tau_2}{\tau_1}\right)^{1/2}\left(\dfrac{1 - \tau_2}{1 - \tau_1}\right)^{\beta}} \qquad (2.16)$$

Following Formulas (2.2), (2.5) and (2.16), numerical computations were carried out for the case when h = 5 m, d_1 = 5 m, $v_0 = a$ = 341.1 m/sec and x = 1/4. The results are displayed in graphical form in Figs. 3 and 4, where $k = d_1/d_2$.

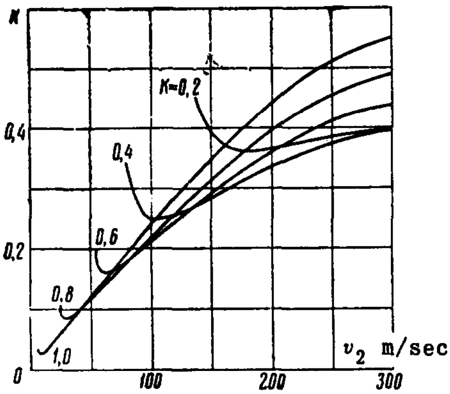

Fig. 3.

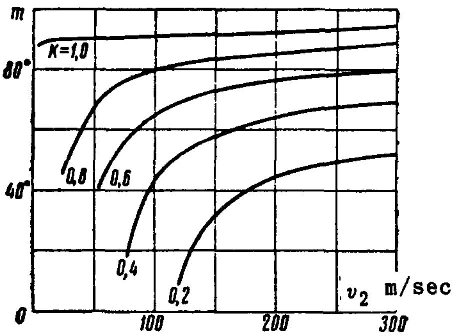

Fig. 4.

3. As was shown by Sedov [14], for any unsymmetrical free jet when $\tau_0 = \tau_s$ at the surface, levelling out of the velocities occurs at a finite distance from the origin of coordinates along a rectilinear segment, beyond which downstream there is established uniform sonic flow. Having regard to Chaplygin's equations

$$\frac{\partial \varphi}{\partial \theta} = \frac{2\tau}{(1-\tau)^\beta} \frac{\partial \psi}{\partial \tau}, \qquad \frac{\partial \varphi}{\partial \tau} = -\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}} \frac{\partial \psi}{\partial \theta}$$

and the differential relation

$$d\xi = \frac{\cos\vartheta}{v} d\varphi - \frac{\sin\vartheta}{(1-\tau)^\beta} d\psi$$

where φ is the velocity potential, we obtain on the surface of the jet

$$d\xi = \frac{2\tau_s}{(1-\tau_s)^\beta} \frac{\cos\vartheta}{a_*} \left(\frac{\partial \psi}{\partial \tau}\right)_{\tau=\tau_s} d\vartheta \tag{3.1}$$

Integrating (3.1) along the segment $[-m, \pi - m]$, we obtain the formula for the length of the segment on which equalization of the velocities occurs:

$$\xi_* = \frac{2d_0}{\pi} \sum_{n=1}^{\infty} \frac{4n}{4n^2-1} \chi_n'(\tau_s) \cos m \tag{3.2}$$

Here $2d_0$ is determined from (2.5).

The results obtained above for efflux from channels with parallel walls can be generalized, as Chaplygin [1] showed to the case when the walls of the channel contain an angle $q\pi (q < 1)$.

In this case, for example, Formula (2.6), which for the sake of simplicity we consider with $d_1 = d_2$ and $m = \pi/2$, takes the form

$$\frac{1}{K} = \frac{2}{\pi} \sin \frac{1}{2} q\pi \left\{ \frac{1}{q} + \Phi(q/2) - q \sum_{n=1}^{\infty} \frac{(-1)^n 4n}{4n^2 - q^2} x_{n/q}(\tau_0) \right\} \qquad (3.3)$$

Here

$$\Phi\left(\tfrac{1}{2} q\right) = \tfrac{1}{2} \left[\beta\left(1 - \tfrac{1}{2} q\right) - \beta\left(1 + \tfrac{1}{2} q\right)\right]$$

$$\beta(x) = \frac{1}{2} \left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right] \qquad \left(\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\right)$$

The function $\beta(x)$ is tabulated [15]. For incompressible fluid (3.3) will have the form

$$\frac{1}{K} = \frac{2}{\pi} \sin \frac{1}{2} q\pi \left\{ \frac{1}{q} + \Phi\left(\frac{q}{2}\right) + q\Psi\left(\frac{q}{2}\right) \right\} \qquad (3.4)$$

$$\Psi\left(\frac{q}{2}\right) = \frac{1}{2} \left[\beta\left(1 - \frac{q}{2}\right) + \beta\left(1 + \frac{q}{2}\right)\right]$$

By virtue of the known relationships [15]

$$\beta(x+1) = \frac{1}{x} - \beta(x), \qquad \beta(1/2) = \frac{\pi}{2}, \qquad \beta(3/2) = 2 - \frac{\pi}{2}$$

we obtain Kirchhoff's formula from (3.4) when $q = 1$.

BIBLIOGRAPHY

1. Chaplygin, S.A., *O gazovykh struiakh (On Gaseous Jets)*. GITTL, 1949.

2. Fal'kovich, S.V., K teorii gazovykh strui (On the theory of gaseous jets). *PMM* Vol. 21, No. 4, pp. 459-464, 1957.

3. Lavrent'ev, M.A., Kumuliativnyi zariad i printsipy ego raboty (Cumulative charge and the principles of its operation). *UMN* Vol. 12, No. 4, pp. 41-56, 1957.

4. Cherry, T.M., Asymptotic expansions for the hypergeometric functions occurring in gas-flow theory. *Proc. Roy. Soc.* Ser. A, Vol. 202, No. 1071, 1950.

5. Slezkin, N.A., Ob udare ploskoi gazovoi strui v bezgranichnuiu stenku (On the collision of a two-dimensional gaseous jet with an infinite wall). *PMM* Vol. 16, No. 2, pp. 227-230, 1952.

6. Troshin, V.I., Obtekanie plastinki struei gaza, vytekaiushchei iz kanala (Flow past a wall by a gaseous jet issuing from a channel). *PMM* Vol. 23, No. 4, pp. 766-769, 1959.

7. Troshin, V.I., Dve zadachi o dozvukovykh gazovykh struiakh (Two problems concerning subsonic gaseous jets). *VMU* Ser. 1, No. 2, pp. 63-64, 1960.

8. Zhukovskii, N.E., Vidoizmenenie metoda Kirkhgofa (Modification of Kirchhoff's method). *Sobr. Soch.* Vol. II, Gostekhizdat, 1949.

9. Frankl', F.I., Asimptoticheskoe razlozhenie funktsii S.A. Chaplygina (Asymptotic expansion of S.A. Chaplygin's functions). *Dokl. Akad. Nauk SSSR* Vol. 58, No. 5, 1947.

10. Truesdell, C., On a function which occurs in the theory of the structure of polymers. *Ann. Math.* Vol. 46, No. 1, 144-157, 1945.

11. Zigmund, A., *Trigonometricheskie riady* (*Trigonometric Series*). GONTI, 131, 1939.

12. Fal'kovich, S.V., Okolozvukovye ploskie techniia gaza s osobymi tochkami na zvukovoi linii (Plane transonic gas flows with singularities on the sonic line). *PMM* Vol. 25, No. 2, pp. 223-224, 1961.

13. Fal'kovich, S.V., Asimptoticheskoe razlozhenie funktsii Chaplygina (Asymptotic expansion of Chaplygin's functions). *Izv. vuzov, ser. matemat.* No. 2(15), pp. 209-212, 1960.

14. Sedov, L.I., *Ploskie zadachi gidrodinamiki i aerodinamiki* (*Plane Problems of Hydrodynamics and Aerodynamics*). Gostekhizdat, 1950.

15. Zyczkowski, M., *Tablice funkcyj Eulera i pokrewnych* (*Tables of Euler Functions and Related Functions*). Panstowe Wydawnictwo Naukowe, Warsaw, 1954.

Translated by A.H.A.